Trigonometric Integrals

- $\sin^2\theta + \cos^2\theta = 1$ $\rightarrow$ $\sin^2\theta = 1 - \cos^2\theta$, $\rightarrow$ $\cos^2\theta = 1 - \sin^2\theta$
- $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$
- $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\boxed{1}$

$$\int \sin^m x \cos^n x \, dx$$

Ex:

odd $3 = 2+1$

① $$\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx$$

$\begin{pmatrix} u = \cos x \\ du = -\sin x \, dx \end{pmatrix}$

$$= \int (1 - \cos^2 x) \sin x \, dx$$

$$= \int (1 - u^2)(-du) = \int (u^2 - 1) \, du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \frac{1}{3}\cos^3 x - \cos x + C$$

② $\int \cos^{2} x \, dx = \int \frac{1}{2}(1+\cos 2x)\,dx$ even

$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$

③ $\int \sin^{4} x \, dx = \int (\sin^2 x)^2 \, dx$ even $4=2\cdot 2$

$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx$

$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right) dx$

$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$

$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$

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④ $\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx$

$= \int (\cos^2 x)^2 \cos x \, dx$

$\begin{pmatrix} u = \sin x \\ du = \cos x \, dx \end{pmatrix}$ $= \int (1 - \sin^2 x)^2 \cos x \, dx$

$$= \int (1-u^2)^2 \, du$$

$$= \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C}$$

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5 $\int \overset{\text{odd}}{\sin^5}(2t) \overset{\text{even}}{\cos^2}(2t) \, dt$

$$= \int (\sin^2(2t))^2 \underline{\sin(2t)} \cos^2(2t) \, dt$$

$$= \int (1-\cos^2(2t))^2 \cos^2(2t) \sin(2t) \, dt \qquad \begin{pmatrix} u = \cos 2t \\ du = -2\sin 2t \, dt \end{pmatrix}$$

$$= \int (1-u^2)^2 u^2 \left(-\frac{1}{2} du\right) = \frac{-1}{2} \int (u^2 - 2u^4 + u^6) \, du$$

$$= -\frac{1}{2}\left(\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right) + C$$

$$= \boxed{-\frac{1}{6}\cos^3 2t + \frac{1}{5}\cos^5 2t - \frac{1}{14}\cos^7 2t + C}$$

$$\sin^2\theta + \cos^2\theta = 1 \;\longrightarrow\; 1 + \cot^2\theta = \csc^2\theta$$

$$\sin^2\theta + \cos^2\theta = 1 \;\longrightarrow\; \tan^2\theta + 1 = \sec^2\theta$$

$$\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$\left(\begin{array}{l} u = \sec x + \tan x \\ du = (\sec x \tan x + \sec^2 x)\,dx \end{array} \right) = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{1}{u} \, du = \ln|u| + C$$

$$\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$$

- $\int \sin nx \cos mx \, dx$
- $\int \sin nx \sin mx \, dx$
- $\int \cos nx \cos nx \, dx$

Fourier

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

ⓐ $\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$ $\leftarrow$

ⓑ $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

ⓒ $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

Ex: $\int \sin 8x \cos 5x \, dx = \int \frac{1}{2}\left(\sin(3x) + \sin(13x)\right) dx$

$$= \frac{1}{2}\left(\frac{-1}{3}\cos 3x - \frac{1}{13}\cos 13x\right) + C$$

"Non-cheaty" way for $\int \sec x \, dx$

$$\int \sec x \, dx = \int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx$$

$\begin{pmatrix} u = \sin x \\ du = \cos x \, dx \end{pmatrix}$ $= \int \frac{\cos x}{1-\sin^2 x} \, dx = \int \frac{1}{1-u^2} \, du$

$\begin{pmatrix} \text{partial} \\ \text{fractions} \end{pmatrix}$ $= \int \frac{1}{(1-u)(1+u)} \, du = \int \left(\frac{1/2}{1-u} + \frac{1/2}{1+u}\right) du$

$$= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C$$

$$= \frac{1}{2}\left(\ln|1+\sin x| - \ln|1-\sin x|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right| + C$$

$$= \cancel{\frac{1}{2}}\ln\left|\frac{1+\sin x}{\cos x}\right|^{\cancel{2}} + C$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$\underbrace{\frac{1}{x} + \frac{1}{x+1}}_{\text{end}} = \frac{x+1}{x(x+1)} + \frac{x}{x(x+1)} = \underbrace{\frac{2x+1}{x(x+1)}}_{\text{start}}$$

$$\left(\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}\right) \cdot x(x+1)$$

$$1 = A(x+1) + Bx = (A+B)x + A$$

$$1 = (A+B)x + A$$

$$\begin{cases} A+B=0 \longrightarrow B = -A = -1 \\ A = 1 \end{cases}$$

$$\frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$$